Twenty Key Ideas in Beginning Calculus

1 Increasing, Decreasing, Instantaneous Rates of Change (scalars)

2 Derivative Functions (expressions)

3 Division by Zero Conundrum

4 Generic Problem Sets
a.) Related Rates
b.) Min/Max

5 Slope of a Tangent to a Curve at a Given Point

6 Fundamental Definitions
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b.) area under a curve

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8 Derivative Theorems for Polynomial Sum, Difference, Product and Quotient

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15 Fundamental Theorem of Calculus, FTC

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17 Successive Approximation Technique to Find
a.) Slope of a tangent to a curve
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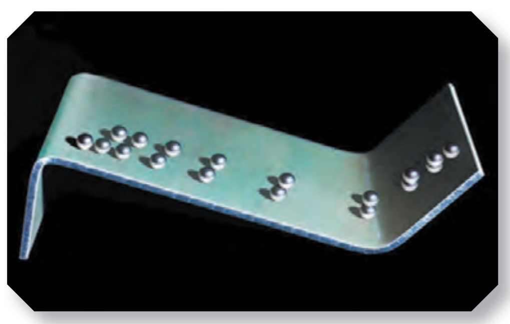
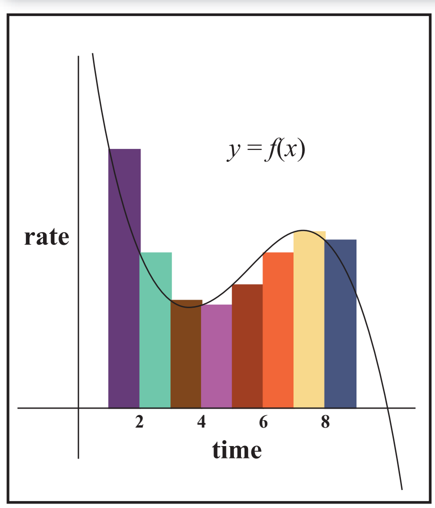
18 Proof

19 Limits, Preconditions for Limits

20 Mean Value Theorem





by Dan Umbarger
www.mathlogarithms.com

Contents

Chapter 1
Using Algebra to Approximate "Instantaneous Speed," Generic Problem Set #1

A major impetus for the development of calculus was 17th-century scientists' need to know about *rates of change* of one quantity compared to another such as the rate of change of position compared to time. *Instantaneous speed* was an especially important quantity to those scientists. Any calculus book you pick up will have dozens, if not hundreds, of references to the terms *rate of change* and *instantaneous rate of change*. Your calculus teacher and your text will assume that you understand what those terms mean. This is not "your father's (or mother's)" calculus text. In fact, as the title clearly states, it is not a text at all, giving the author license to do many things differently.

Both of these terms, *rate of change* and *instantaneous speed*, are highly abstract and could only be imagined at the time Isaac Newton and Gottfried Leibniz were putting the final touches on what we now call calculus. The development of strobe light photography allows us to see photographs that can help us understand by inference the meaning of both of those terms.

Photograph by Terence Kearey, Sweden

In the image at left, you see a ball rolling down an incline. There is one ball whose image is repeatedly captured every time the strobe light flashes. Notice that the ball images at the top of the incline are closer together than those in the middle and much closer together than the ball images toward the bottom. Because the strobe light flashes at equal time intervals, you can infer that the ball picks up speed over time as the ball's inertia is overcome by gravity. The *speed* of the ball is *increasing* over time.

The same comments can be made about the image at right. The interval between the images of the ball increase as it descends the ramp, indicating an *increasing speed*, until the ball reaches the bottom of the ramp and starts up the other side. There, the images start occurring more closely spaced, indicating a *decreasing speed*. Then, the ball makes the return trip and stops just below where it started because some energy was lost to friction during the trip.

Photograph by Terence Kearey, Sweden

Photograph by Terence Kearey, Sweden

The figure at left also demonstrates both *increasing* and *decreasing speeds*. The weight on the end of the string is photographed closely together at first but further apart at the bottom of its pendulum swing, indicating an *increase of speed*. Correspondingly, the weight's image is captured closer and closer together as the weight approaches the end of its pendulum swing, indicating a *decrease of speed*.

With less commentary, three more images are shown below. A large gap between images indicates a (relatively) high speed while a small gap between images indicates a (relatively) low speed. The two images at left have an initial external force acting on them in addition to gravity, while the image of the falling egg has only gravity acting on it.

Decreasing then Increasing Speed

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Decreasing Speed

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Increasing Speed

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Instantaneous Speed

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Over and over, the terms *increasing* and *decreasing speeds* have been used. Another term that is used in beginning calculus books is *instantaneous speed*. The image at the right gives an idea of what the term *instantaneous speed* must mean. (Notice the rifling marks on the bullet. It may be tempting here to assume from this image that the bullet is not moving hence has a speed of zero. The next several pages will dispel such a notion.)

Changing Rates of Speed

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Instantaneous Speed

This image was taken the exact instant a Prandtl-Glauert condensation cloud formed about a jet.
Christopher Pasatieri/Reuters via National Geographic News
news.nationalgeographic.com/news/2009/06/photogalleries/week-in-news-pictures-133/photo2.html

PROBLEM 1: A missile is fired at a target 64 miles away. The distance in miles that the missile has traveled from its starting point is given by the function $f(t) = t^2$, where t is in minutes. Determine the speed of the missile at time $t = 8$, the precise instant when it strikes the target—its instantaneous speed. *Here the function $f(t) = t^2$ relates two values, time and distance, with distance depending on the value of time.*

As the term *instantaneous speed* is unfamiliar to us, we engage in a classic problem-solving technique. *Look for something that is similar to what you wish to know, think about the similarities and differences of the two situations and then try to use knowledge or results from the known situation to apply to the new unknown one* ... a primitive form of induction.

From work in beginning algebra, you should have seen or heard the term speed. (Sometimes you see the word velocity.) It is implied in the formula $d = r \times t$ ... distance = rate × time. For example, 240 miles = 60 mph × 4 hours. The term *rate* in this formula is understood to be *average speed*. *Average speed* means speed over a specified interval of time ... in this case, one hour. That's where the *ph* comes from in the *mph*, miles per hour. If you travel 100 miles in two hours you are traveling, on average, 50 mph, regardless of how fast you were going at each minute during those two hours.

Solving $d = r \times t$ for the variable r using algebra, we should be able to find the average speed of the missile in Problem 1 using the formula $r = \frac{d}{t}$ over different time intervals (such as from one minute into the flight to eight minutes into the flight, etc.). The following table presents the calculations with the ending time of each interval being when the missile reaches the target. Therefore, the ending position is always 64 miles.

Time in minutes (domain)		Position in miles, $d = t^2$ (range)		$r = \frac{d}{t}$
Starting	Ending	Starting	Ending	speed = $\frac{\text{ending position}-\text{starting position}}{\text{ending time}-\text{starting time}}$
0	8	0	64	$r = \frac{64-0}{8-0} = \mathbf{8}$ mpm
1	8	1	64	$r = \frac{64-1}{8-1} = \mathbf{9}$ mpm
2	8	4	64	$r = \frac{64-4}{8-2} = \mathbf{10}$ mpm
3	8	9	64	$r = \frac{64-9}{8-3} = \mathbf{11}$ mpm
4	8	16	64	$r = \frac{64-16}{8-4} = \mathbf{12}$ mpm
5	8	25	64	$r = \frac{64-25}{8-5} = \mathbf{13}$ mpm
6	8	36	64	$r = \frac{64-36}{8-6} = \mathbf{14}$ mpm
7	8	49	64	$r = \frac{64-49}{8-7} = \mathbf{15}$ mpm
8	8	64	64	$r = \frac{64-64}{8-8} = \mathbf{16}$ mpm ??? **Indeterminate division by zero!!!**

The sequence of average speeds calculated above, 8 mpm, 9 mpm, 10 mpm, 11 mpm, 12 mpm, 13 mpm, 14 mpm and 15 mpm, seems reasonable as we know from watching tv and movies that rocket ships get faster and faster as the initial stationary inertia of the projectile is overcome by the rocket’s thrust. Did we achieve our goal of obtaining the instantaneous speed of the rocket at $t = 8$ or equivalently at $d = 64$? Well, no, but we suspect that it must be greater than 15 mpm.

Let us continue this same pattern of analysis. Let’s concentrate on the average speed as the missile approaches its target over the last minute ($t = 7$ to $t = 8$) of its flight: $f(t) = t^2$.

Time in minutes (domain)		Position in miles, $d = t^2$ (range)		$r = \frac{d}{t}$
Starting	Ending	Starting	Ending	speed = $\frac{\text{ending position} - \text{starting position}}{\text{ending time} - \text{starting time}}$
7	8	49	64	$r = \frac{64-49}{8-7}$ = **15** mpm
7.9	8	62.41	64	$r = \frac{64-62.41}{8-7.9}$ = **15.9** mpm
7.99	8	63.8401	64	$r = \frac{64-63.8401}{8-7.99}$ = **15.99** mpm
7.999	8	63.984001	64	$r = \frac{64-63.984001}{8-7.999}$ = **15.999** mpm
8	8	64	64	$r = \frac{64-64}{8-8}$ = **16** mpm ??? **Indeterminate division by zero!!!**

Combining the old data for average speeds as the missile approached its target with the new data for average speeds, we now get the sequence of “average speeds” as calculated above for decreasing intervals of time:

8, 10, 12, 14, 15, 15.9, 15.99, and 15.999

From the pattern of average speeds shown above, it would seem reasonable to conclude that the “instantaneous speed” when $t = 8$ minutes and $d = 64$ miles is 16 mpm. However, how could you justify or prove such an answer? The answer is, “Using simple algebraic skills you can’t.” You can only approximate. (See Appendix A, $15.999\ldots = 16$, for an interesting discussion of this.) The algebraic attempt to calculate the instantaneous speed using the formula for average speed over a time interval of zero length results in a division by zero. The proof that the instantaneous speed of the missile at $t = 8$ seconds and $d = 64$ miles is exactly 16 mpm can only be obtained using knowledge and skills and vocabulary learned in calculus.

OK let's review.

PROBLEM 1: A missile is fired at a target 64 miles away. The distance in miles that the missile has traveled from its starting point is given by the function $f(t) = t^2$, where t is in minutes. Determine the speed of the missile at time $t = 8$, the precise instant when it strikes the target—its instantaneous speed.

The sequence of "average speeds" as calculated above for decreasing intervals of time is:

8, 10, 12, 14, 15, 15.9, 15.99, and 15.999 . . . 16 mpm ???

We suspect that the "instantaneous speed" for the missile when $t = 8$ minutes and $d = 64$ miles is 16 mpm but cannot really prove our suspicions. Look at the sequence of average speeds again. How would you describe them? Are they getting larger with each term? Yes. Could an average speed ever get to be 100 mpm? Do you think that given the function $f(t) = t^2$ and the time and distance constraints ($0 <= t <= 8$ minutes, $0 <= d <= 64$ miles) that the missile will ever speed up to 100 mpm in its last 0.001 minute of flight? Do you think that the instantaneous speed will ever get larger than 16? Do you think that the instantaneous speed will ever reach 16? Let's look at those numbers one more time:

(average speeds over decreasing time intervals)

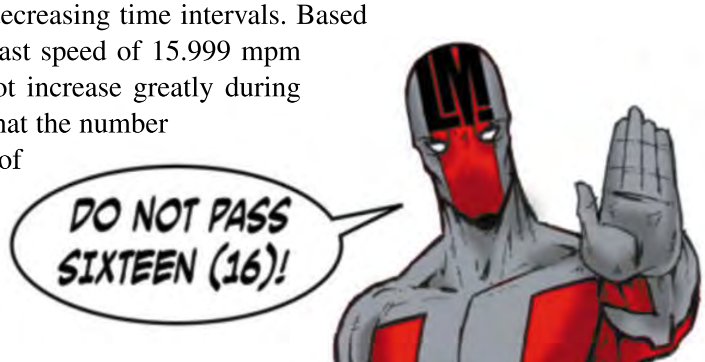
8, 10, 12, 14, 15, 15.9, 15.99, 15.999 mpm

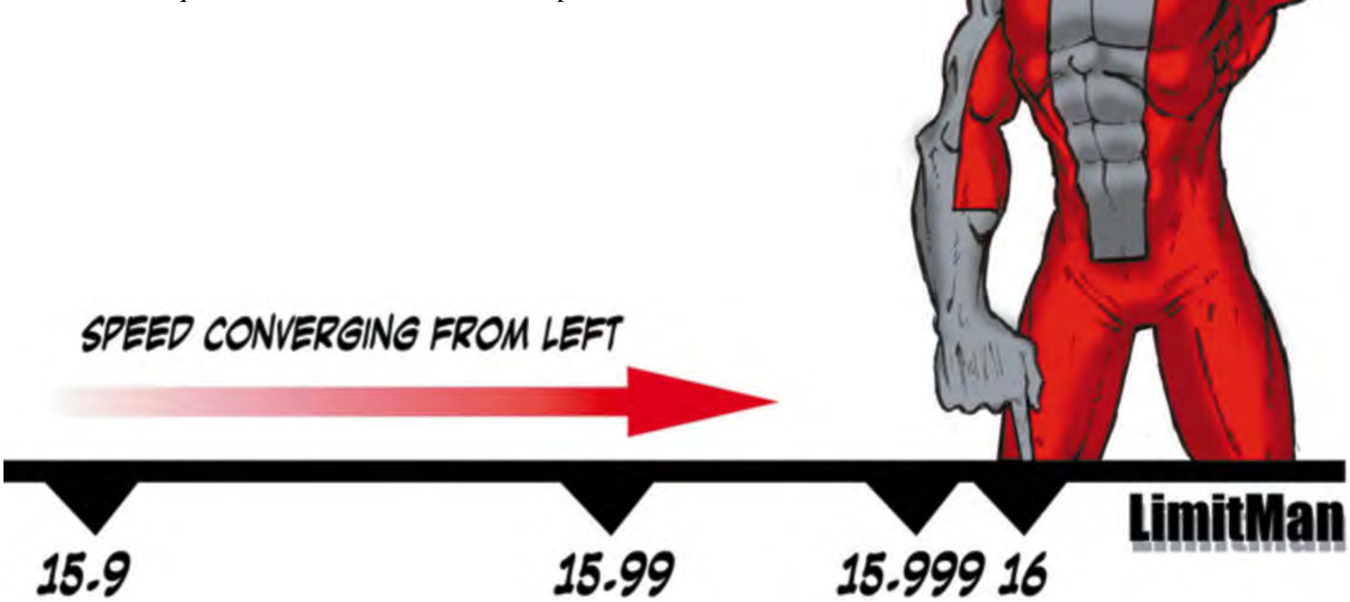
$\hat{2}$ $\hat{2}$ $\hat{2}$ $\hat{1}$ $0.\hat{9}$ $0.0\hat{9}$ $0.\hat{0}09$

(increase of average speed from the previous time interval)

After one month of instruction, a calculus student should be able to solve for the exact answer to Problem 1 inside the space of this text box and be able to do so in less than a minute.

These numbers (average speeds) are increasing each for each interval, but the rate of increase each time seems to be decreasing with the decreasing time intervals. Based on the pattern of speed increases shown above, the last speed of 15.999 mpm over the last 0.001 minute of flight will probably not increase greatly during the remaining time of the missile's flight. It appears that the number 16 acts as a sort of barrier or limit to the progression of numbers we are calling average speeds. The study of calculus will give you the tools to justify this suspicion. It will allow you to say, without any doubt, that the instantaneous speed of the missile in this question at $t = 8$ minutes is 16 mpm.

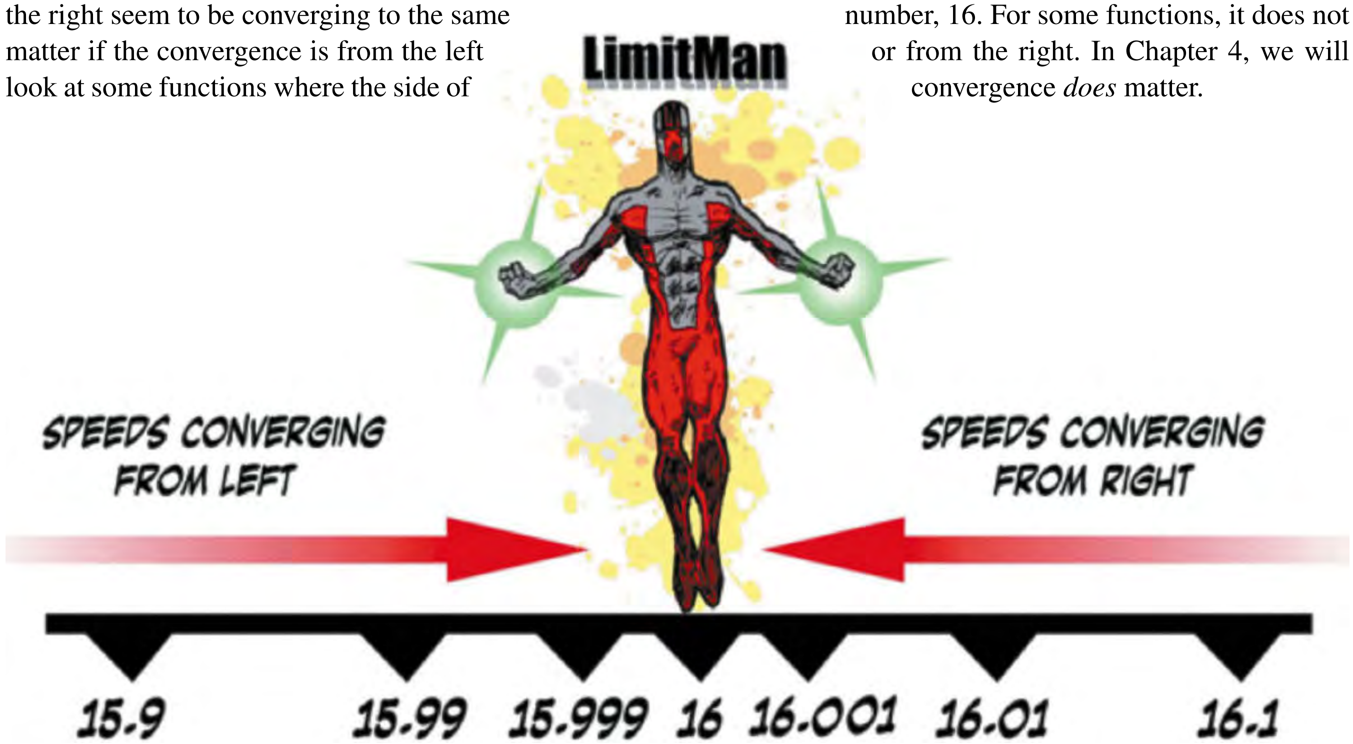




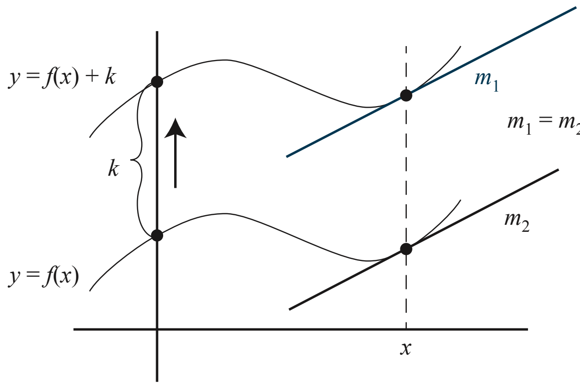
Would it matter if you were to calculate average speeds from the left of t = 8 minutes (from t = 7) or from the right (from t = 9)? Well, obviously in the real world, the missile cannot approach t = 8 minutes from the right, even if it missed the target (unless you ran video in reverse). Mathematically it should not matter.

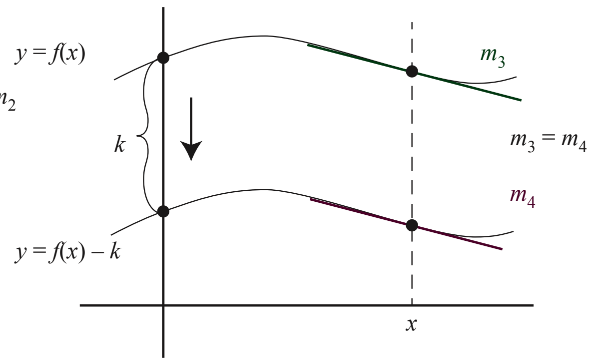
Time in minutes (domain)		Position in miles, $d = t^2$ (range)		$r = \frac{d}{t}$
Starting	Ending	Starting	Ending	speed = $\frac{\text{ending position}-\text{starting position}}{\text{ending time}-\text{starting time}}$
↓ 7	8	↓ 49	64	$r = \frac{64-49}{8-7}$ = **15** mpm ↓
↓ 7.9	8	↓ 62.41	64	$r = \frac{64-62.41}{8-7.9}$ = **15.9** mpm ↓
↓ 7.99	8	↓ 63.8401	64	$r = \frac{64-63.8401}{8-7.99}$ = **15.99** mpm ↓
↓ 7.999	8	↓ 63.984001	64	$r = \frac{64-63.984001}{8-7.999}$ = **15.999** mpm ↓
8	8	64	64	$r = \frac{64-64}{8-8}$ = **16** mpm ??? **Indeterminate division by zero!!!**
↑ 8.001	8	↑ 64.016001	64	$r = \frac{64-64.016001}{8-8.001}$ = **16.001** mpm ↑
↑ 8.01	8	↑ 64.1601	64	$r = \frac{64-64.1601}{8-8.01}$ = **16.01** mpm ↑
↑ 8.1	8	↑ 65.61	64	$r = \frac{64-65.61}{8-8.1}$ = **16.1** mpm ↑
↑ 9	8	↑ 81	64	$r = \frac{64-81}{8-9}$ = **17** mpm ↑

Here both the average speed as t approaches 8 from the left and the average speed as t approaches 8 from the right seem to be converging to the same number, 16. For some functions, it does not matter if the convergence is from the left or from the right. In Chapter 4, we will look at some functions where the side of convergence *does* matter.

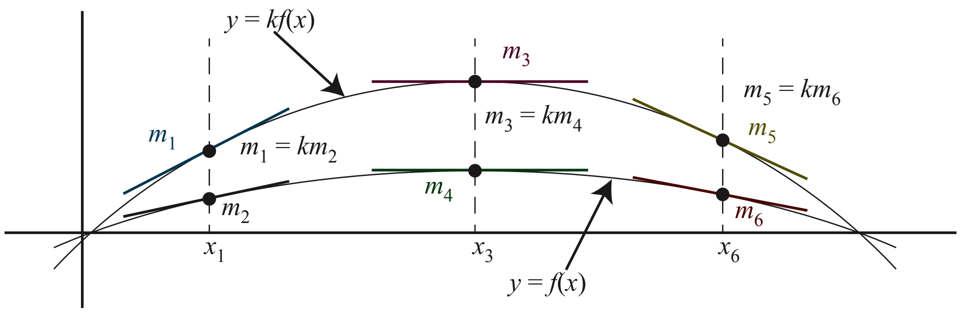


A function $f(x)$ can be vertically shifted by k, giving $f(x) + k$. The slopes of the tangents to function $f(x)$ and the new function $f(x) + k$ at any point x will be the same: $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x) + k]$. This is true whether the vertical shift is up or down: $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x) - k]$.

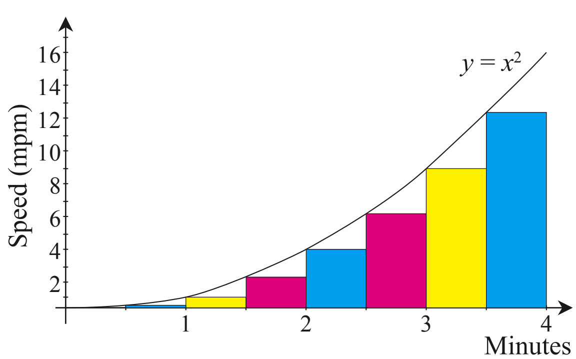




When a function $f(x)$ is multiplied by a constant k, $k[f(x)]$, the slope of the tangent to the function, $k[f(x)]$, will be k times the slope of the tangent to the function $f(x)$.



At any point x, the slope of the tangent to the curve $k[f(x)]$ is k times the slope of the tangent to the curve $f(x)$: $\frac{\mathrm{d}}{\mathrm{d}x}[kf(x)] = k\left\{\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]\right\}$.

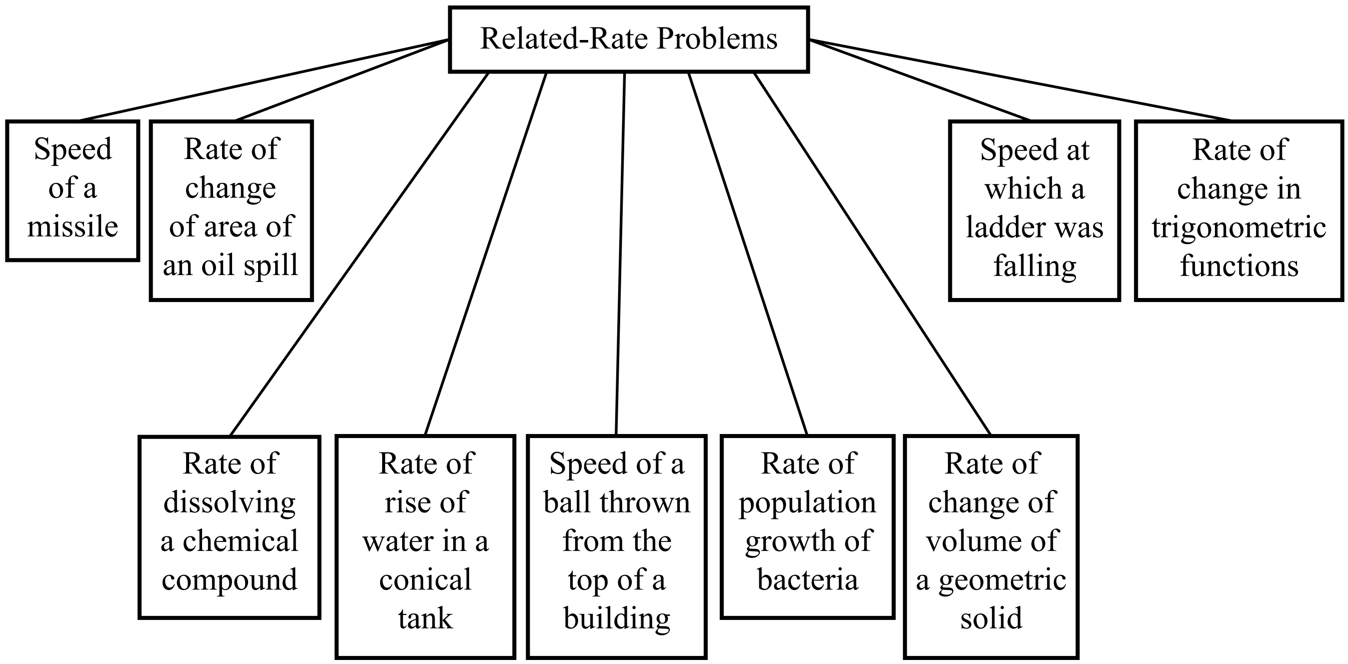


"I'd like to be your derivative so I could lay next to your curves."
The Big Bang Theory, a tv show about geeky, socially awkward scientists.

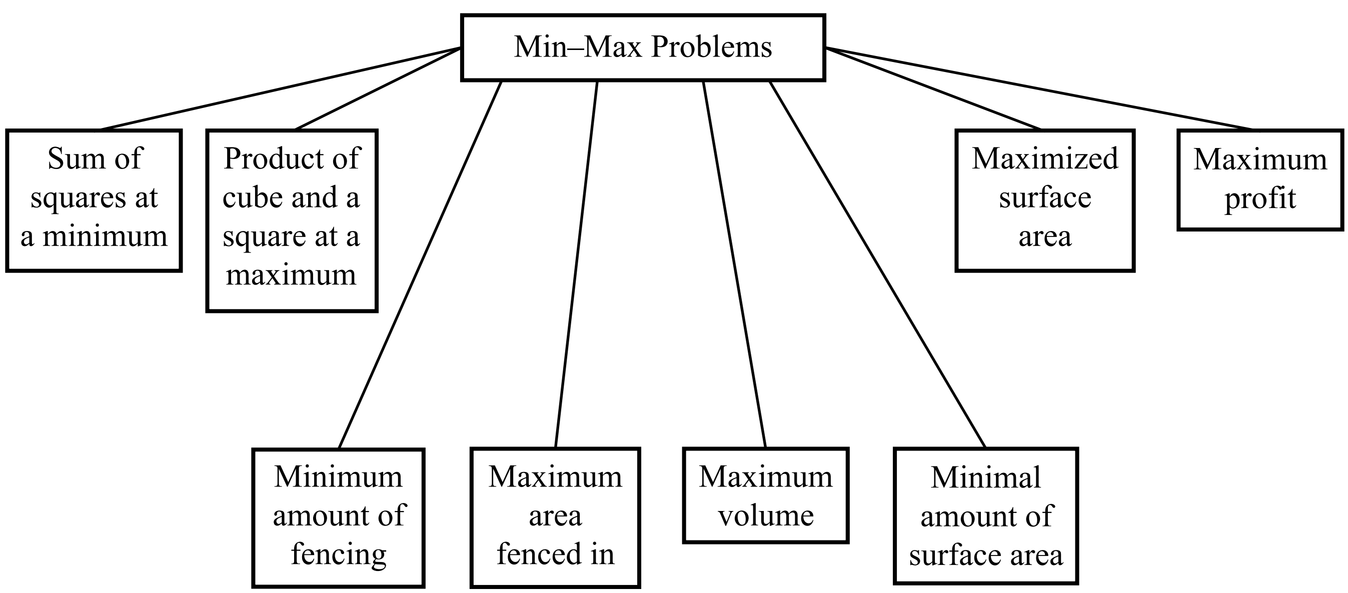
"I'd like to be your integral so I could fill in your spaces."
The Big Bang Theory, a tv show about geeky, socially awkward scientists.

Chapter 10 Review

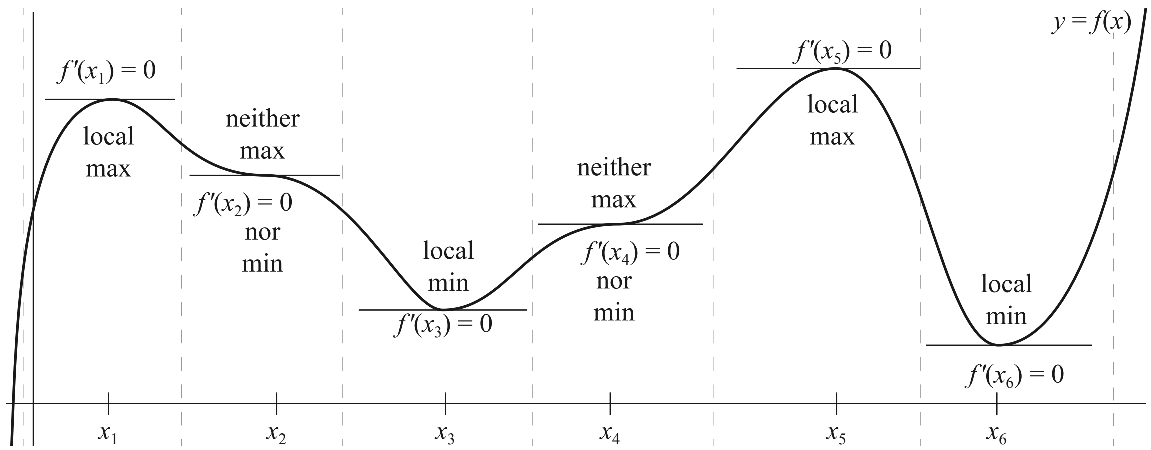
In Chapter 1, we learned about the idea of *generic problem sets*. That is, by learning to recognize that a problem was in a certain category and by knowing that there is a general approach to every problem in that category, you reduce your learning curve and work effort by several orders of magnitude.



Chapter 10 showed another example of a generic problem set: All of the problems share a commonality that allowed us to solve them all with the same general approach. It would be difficult to overstate the importance of this idea!



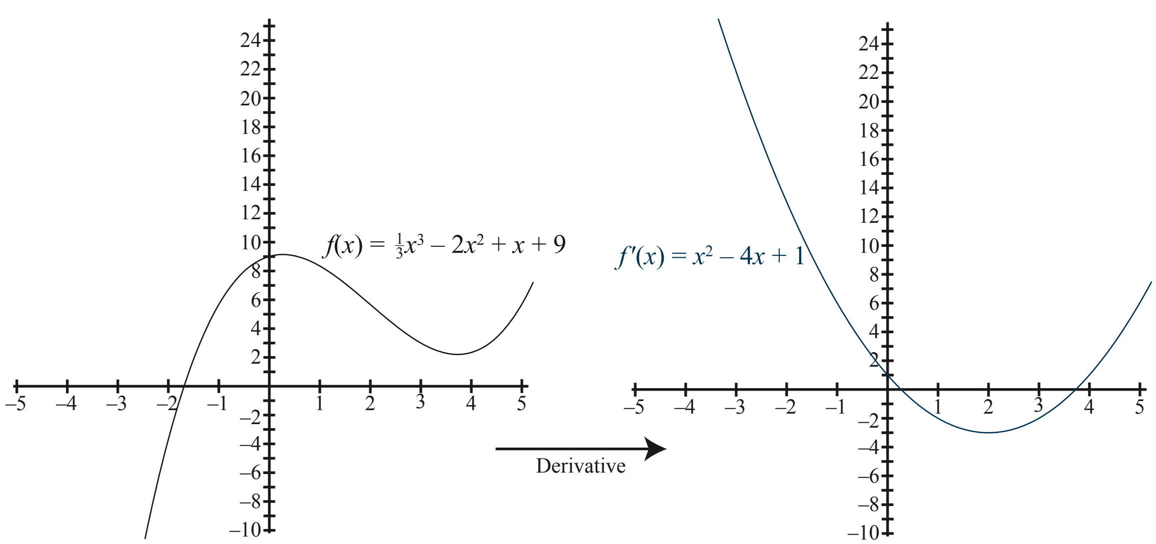
On the lead-in page to Chapter 10, we saw this nifty figure courtesy of my editor. It shows that whenever a function is at a local minimum or maximum, the derivative at that x value is equal to 0.



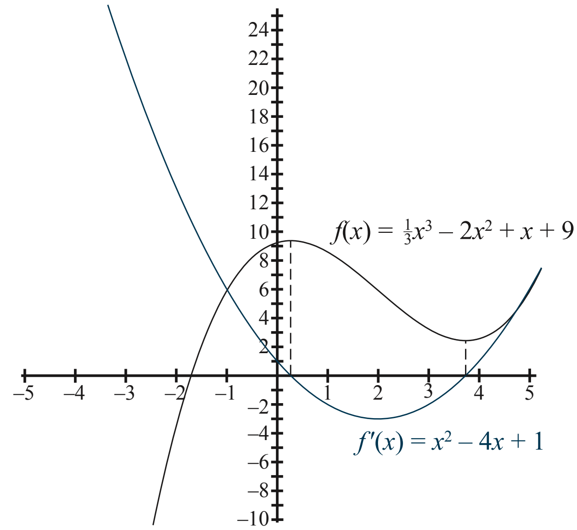
Another way to look at this information is to remember that the term "derivative" is actually a short way of saying "derivative function." You may remember in Chapter 1 that a derivative was referred to as a "derivative (function)."

$f(x) = x^n$	$f'(x) = nx^{n-1}$
$f(x) = \dfrac{1}{3}x^3 - 2x^2 + x + 9$	$f'(x) = \dfrac{1}{3}\left(3x^{3-1}\right) - 2\left(2x^{2-1}\right) + 1\left(x^{1-1}\right) + 0$ $= x^2 - 4x + 1$

Below we see the two functions $f(x)$ and $f'(x)$ graphed side by side.

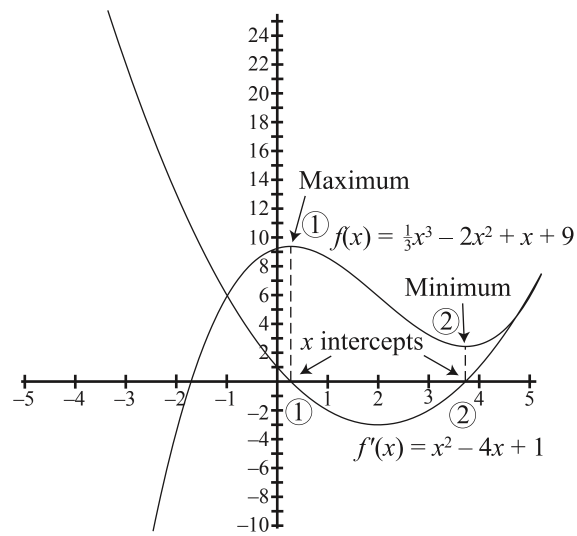


Finally, we see the two functions $f(x)$ and $f'(x)$ superimposed on the same axes.



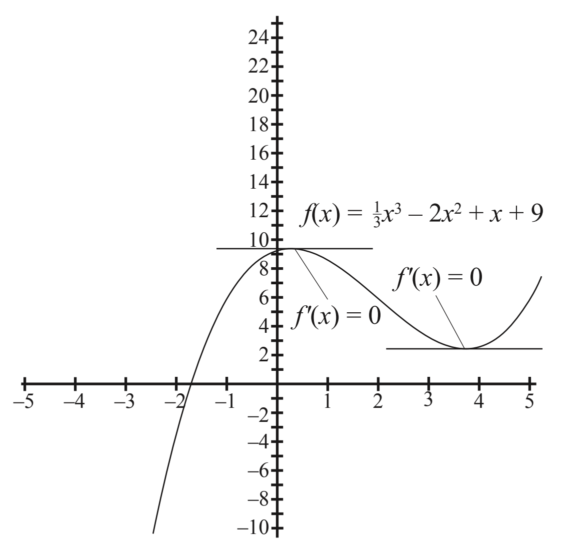
$f'(x) = 0$ when $f(x)$ is at a local max or min.

The x-intercept for a function occurs whenever the function crosses the x-axis—when the y value equals 0. The x-intercepts of the derivative (function), $f'(x) = x^2 - 4x + 1$, are shown at right as Points 1 and 2. Notice that the function $f(x)$ is at a local maximum at Point 1 and at a local minimum at Point 2.



$f'(x) = 0$ when $f(x)$ is at a local max or min.

This was taught previously using a graphic comparable to the the graph at right.



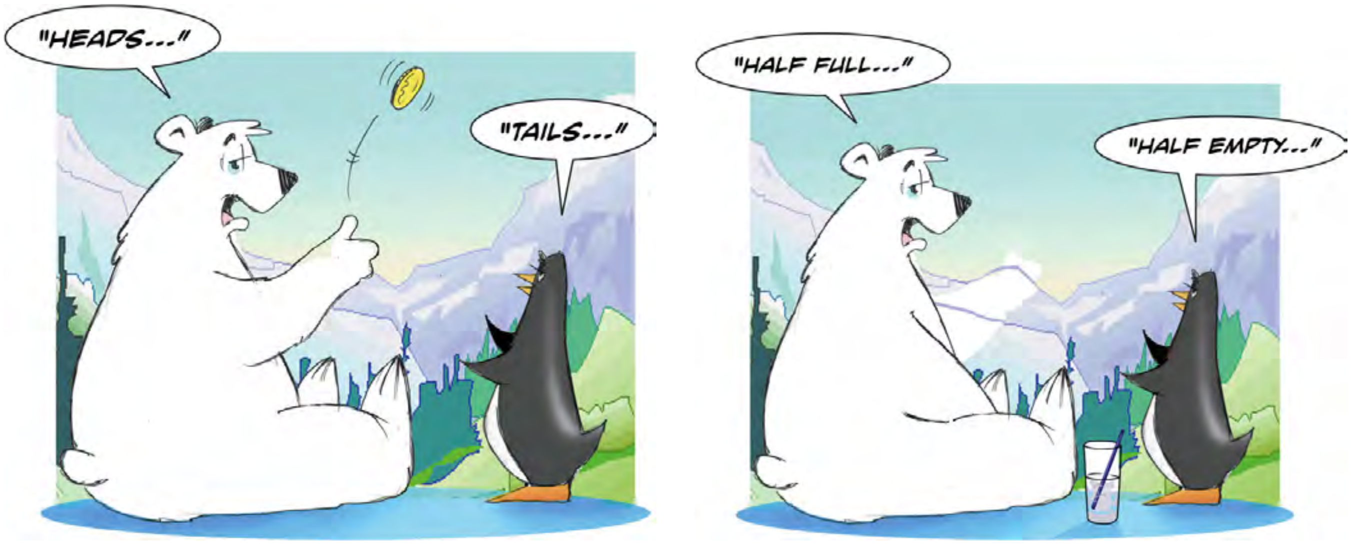
$f'(x) = 0$ when $f(x)$ is at a local max or min.

Chapter 11 Lead In

In calculus, it is frequently the case that there are two different ways of looking at the same information. This was shown in Chapter 2 when we found that "instantaneous speed" was mathematically equivalent to "slope of a tangent to a curve at a given point." In Chapter 11, we will again be seeing that the same information can be viewed from two different perspectives. The ability to see the same idea from two different perspectives is very, very helpful in the study of calculus. This ability comes so naturally to your calculus teacher that he or she may not point this out to you. He or she will assume that if you understand one of two equivalent ideas that you understand the other.

Chapter 2 emphasized an important concept: For functions of the form $y = f(x)$, an average speed between two points in time, $(a, f(a))$ and $(b, f(b))$, could be obtained using the formula $\frac{f(b)-f(a)}{b-a}$. Alternatively, the slope of a secant line between two points, (x_1, y_1) and (x_2, y_2), on a function could be found using the formula $m = \frac{y_2-y_1}{x_2-x_1}$. Although these were two different interpretations of the data, they were both mathematically equivalent. *This equivalence allowed us to interchange ideas between a conceptual (real-world) problem and a graph.* Chapter 4 presented a definition, "The Definition of $f'(x)$," that allowed us to decrease the interval between the two points in time (or the two points used to determine secant slope) down to zero, allowing us to find either the "instantaneous speed" or the "slope of a tangent" at a given point. Either process could be referred to by the generic term "finding a derivative."

All the problems in Chapters 1–10 involved rates of change and were representative of the branch of calculus known as differential calculus. Chapter 11 begins the study of a different branch of calculus, integral calculus. As with differential calculus, the doorway to this branch of calculus will be a definition, The Definition of Area Under a Curve. Again, as with differential calculus, there will be a mathematical equivalence between a real-world concept and a mathematical one. *This equivalence will again allow us to interchange ideas between a conceptual (real-world) problem and a graph.*



Two different ways of looking at the same idea.

Specifically, in Chapters 11 and 12, we will find that the area under a curve is equivalent to finding how much "distance was traveled," "pressure is exerted," "work occurred," etc.

1.) A rocket is accelerating (speeding up). Its speed (in mpm) is given by the function $r = t^2$. **How far did the rocket travel in the first four minutes?** In general this problem is worked using the formula Distance = rate × time or $d = rt$.

2.) Given that the density of water is 62.5 lb/ft^3, **find the force of the water against a triangular dam 50 feet wide and 40 feet deep.** In general, this problem is worked using the formula pressure = force × area or $p = fa$.

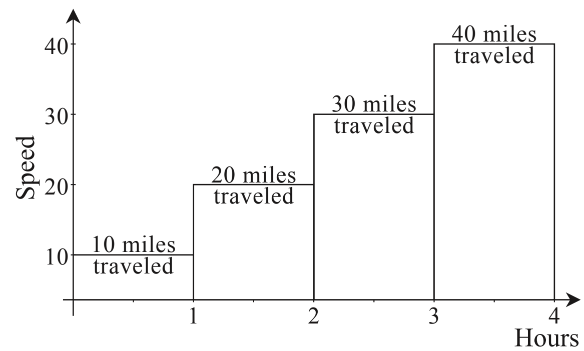
3.) A force of 500 pounds compresses a spring 2 inches from its natural length of 16 inches. **Use Hooke's Law, $f = kd$, to find the work done in compressing the spring an additional four inches.** In general, this problem is worked using the formula Work = Force × Distance or $w = fd$.

Chapter 11

Using Algebra to Introduce Integral Calculus

A car travels	10 mph for one hour
	20 mph for one hour
	30 mph for one hour
and	40 mph for one hour
What was the total distance that the car traveled?	

Total distance the car traveled = $d_1 + d_2 + d_3 + d_4$. This total distance can be thought of in two different ways as shown below.



(Using the distance formula, $d = rt$)

$$d_{\text{hour 1}} + d_{\text{hour 2}} + d_{\text{hour 3}} + d_{\text{hour 4}} =$$

Note that distance is rate times time: $d = rt$

$$(r_1t_1) + (r_2t_2) + (r_3t_3) + (r_4t_4) =$$

$$(10 \times 1) + (20 \times 1) + (30 \times 1) + (40 \times 1) =$$

$$10 + 20 + 30 + 40 = 100 \text{ miles}$$

(Using the area of a rectangle formula, $a = bh$)

$$d_{\text{hour 1}} + d_{\text{hour 2}} + d_{\text{hour 3}} + d_{\text{hour 4}} =$$

Note that area is base times height: $d = a = bh$

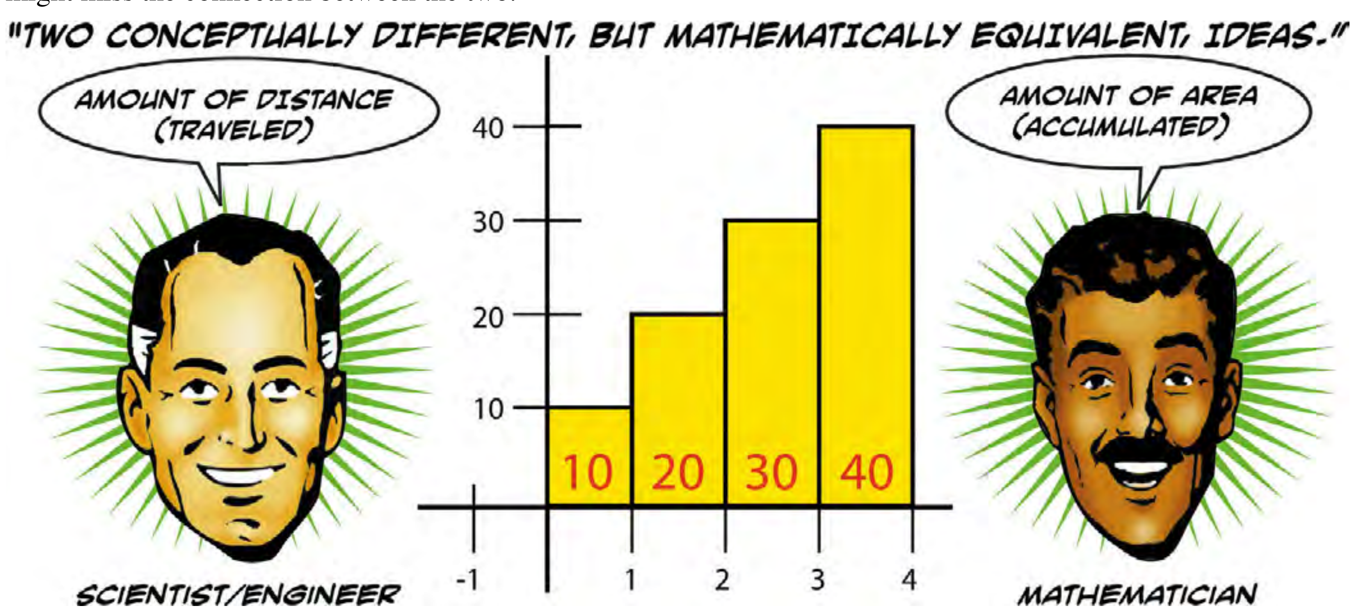
$$(b_1h_1) + (b_2h_2) + (b_3h_3) + (b_4h_4) =$$

$$(10 \times 1) + (20 \times 1) + (30 \times 1) + (40 \times 1) =$$

$$10 + 20 + 30 + 40 = 100 \text{ miles}$$

It is more than just a little bit important for you to understand what is demonstrated above. The thought process at the left uses the formula $d = rt$ and shows how a science or engineering teacher thinks. The thought process at the right uses the formula $a = bh$ and shows how a mathematics teacher thinks. These two ways of thinking about the problem are mathematically equivalent: The width of the base of each rectangle is equal to the time the car spends traveling at each speed. Correspondingly, the height of each rectangle is equal to the rate at which the car was traveling.

Math teachers teach in generalities using abstractions assuming that their students understand that such skills can be applied to specific problems in business, science, engineering, psychology, and many other fields. But the generalities that math teachers use are sometimes very far removed from any application, so far that the student might miss the connection between the two.



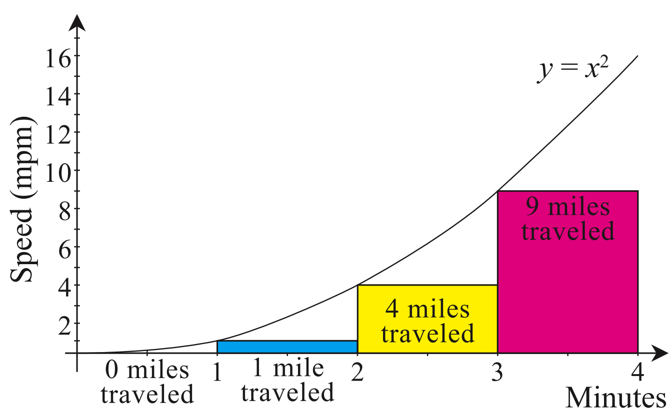
A rocket is accelerating (speeding up). Its speed (in mpm) is given by the function $r = t^2$. *How far did the rocket travel in the first 4 minutes?*

$$r = t^2$$

Time (minutes)	Speed (mpm)
0	0
1	1
2	4
3	9
4	16

This new problem is pretty well the same as the one before except for the fact that the object in motion is now moving with a continuous acceleration. Before, because the car's speed over each of the equally spaced time intervals was constant and because the change in speed was effectively instantaneous each time the speed did change, we could represent the distance traveled over each time interval as the area of a rectangle and calculate the respective areas (distance traveled over each time interval). The fact that this new problem is conceptually the same as the previous one means that the answer to the new problem could be found by finding the area under the curve from $x = 0$ to $x = 4$. However, this time we have a continuous and variable speed, resulting in a curved line which will not allow us to form and add up areas/distances using the formulas $d = rt$ and $a = bh$. Conceptually, there are similarities here to Chapters 1 and 2 in this book. There, we did not know how to find instantaneous speed or slope of a tangent to a curve at a given point, so we used our algebra skills to generate successive approximations of the desired information, $r = \frac{d}{t}$ and $m = \frac{y_2 - y_1}{x_2 - x_1}$. That gives us the idea of successively approximating the area under the curve by adding up an increasing number of smaller and smaller rectangle areas.

As has been shown on the previous page, each of the four rectangle areas shown here approximates the distance the rocket traveled over that time interval. (Author's note, partition 0 is from 0 to 1. Note also that the y axis is compressed in the graph below.)



The distance that the rocket traveled over four minutes is approximately 14 miles (0 + 1 + 4 + 9). Notice that the rectangles do not completely fill the space below the curve. There are four rounded triangles missing from the calculated area below the curve $y = x^2$. That means that this is a low estimate.

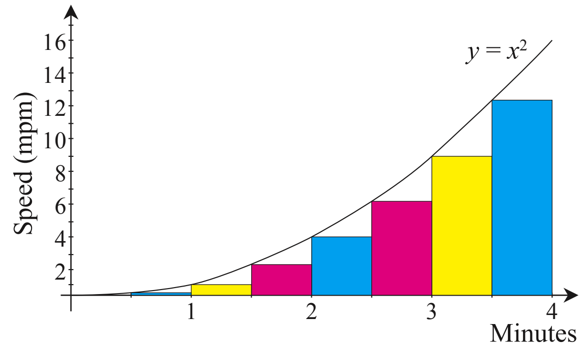
x	Rectangle height x^2	Rectangle base (interval/number of rectangles)	Area of the rectangle $a = bh$
0	0	1	0
1	1	1	1
2	4	1	4
3	9	1	9

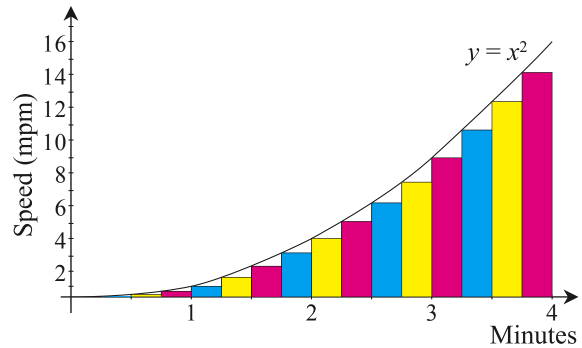
To improve the previous approximation, we next increase the number of rectangular partitions to eight. Now the base of each rectangle is the size of the interval divided by the number of partitions = $\frac{4}{8} = \frac{1}{2}$. (Author's note, partition 0 is from 0 to $\frac{1}{2}$. Note also that the y axis is compressed.) The sum of these eight rectangle areas is $0 + \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{16}{8} + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} = \frac{140}{8} = 17.5$ units.

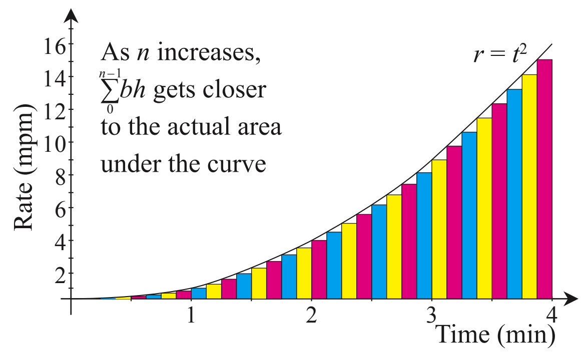
x	Rectangle height x^2	Rectangle base $\frac{\text{interval}}{\text{number of rectangles}}$	Area of the rectangle $a = bh$
0	0	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$
1	$1 = \frac{4}{4}$	$\frac{1}{2}$	$\frac{1}{2} = \frac{4}{8}$
$\frac{3}{2}$	$\frac{9}{4}$	$\frac{1}{2}$	$\frac{9}{8}$
2	$4 = \frac{16}{4}$	$\frac{1}{2}$	$2 = \frac{16}{8}$
$\frac{5}{2}$	$\frac{25}{4}$	$\frac{1}{2}$	$\frac{25}{8}$
3	$9 = \frac{36}{4}$	$\frac{1}{2}$	$\frac{9}{2} = \frac{36}{8}$
$\frac{7}{2}$	$\frac{49}{4}$	$\frac{1}{2}$	$\frac{49}{8}$

Let's talk about those units for a moment. The base of the rectangle is minutes (min), and the height of the rectangle is miles per minute $\left(\frac{\text{miles}}{\text{min}}\right)$. So, the product is $\frac{\text{miles}}{\cancel{\text{min}}} \times \cancel{\text{min}} =$ miles. That's good; the units match what we thought we were calculating.

Therefore, the distance the rocket traveled over four minutes is approximately 17.5 miles. Continuing this pattern for 16 and 32 rectangles, we get the figures at right.







The mathematics of adding up these areas is getting more laborious and time consuming. Perhaps there is a pattern we could use. Using the data from the table, let's sum up the areas of the graph with eight rectangles.

$$\begin{aligned}
\text{Area} &= 0 + \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{16}{8} + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} = \frac{140}{8} = 17.5 \\
&= \text{area}_0 + \text{area}_1 + \text{area}_2 + \text{area}_3 + \text{area}_4 + \text{area}_5 + \text{area}_6 + \text{area}_7 = \text{total area} \\
&= b_0h_0 + b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4 + b_5h_5 + b_6h_6 + b_7h_7
\end{aligned}$$

Now, since the base of all these rectangles is the same—$\frac{\text{interval length}}{\text{number of rectangles}}$—substitute b for b_n:

$$\begin{aligned}
\text{Area} &= bh_0 + bh_1 + bh_2 + bh_3 + bh_4 + bh_5 + bh_6 + bh_7 \\
&= b \times (h_0 + h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7), \quad \text{factor out } b \\
&= \frac{1}{2} \times \left[0^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{6}{2}\right)^2 + \left(\frac{7}{2}\right)^2\right] \\
&= \frac{1}{2} \times \left[0 + \frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4} + \frac{25}{4} + \frac{36}{4} + \frac{49}{4}\right] = \frac{1}{2} \times \frac{1}{4} \times [0 + 1 + 4 + 9 + 16 + 25 + 36 + 49], \quad \text{factor out } \frac{1}{4} \\
&= \frac{1}{8} \times \left[0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2\right] = \frac{1}{8} \times \text{the sum of the squares of the integers from 0 to 7} \\
&= \frac{1}{8} \times \frac{7 \times 8 \times 15}{6} = 17.5, \quad \text{a formula from precal states } \sum_0^{n-1} i^2 = \frac{(n-1)(n)(2n-1)}{6}, \text{ see Appendix H}
\end{aligned}$$

In general, this work suggests that the area of any n rectangles could be found by the formula $b^3 \times \frac{(n-1)(n)(2n-1)}{6} = \left(\frac{\text{interval length}}{\text{number of rectangles}}\right)^3 \times \frac{(n-1)(n)(2n-1)}{6}$. Note that this only works to approximate the area under the curve $y = x^2$ on the interval from zero to four.

$$\text{16 rectangles, on } [0,4]: \left(\frac{4}{16}\right)^3 \times \frac{15 \times 16 \times 31}{6} = \frac{15 \times 16 \times 31}{64 \times 6} = 19.375$$

$$\text{32 rectangles, on } [0,4]: \left(\frac{4}{32}\right)^3 \times \frac{31 \times 32 \times 63}{6} = \frac{31 \times 32 \times 63}{512 \times 6} = 20.34375$$

$$\text{64 rectangles, on } [0,4]: \left(\frac{4}{64}\right)^3 \times \frac{63 \times 64 \times 127}{6} = \frac{63 \times 64 \times 127}{4{,}096 \times 6} = 20.8359375$$

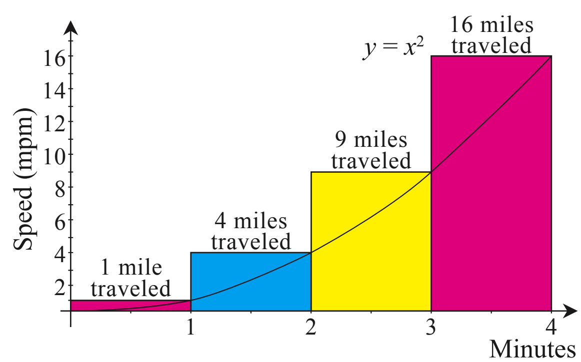
$$\text{128 rectangles, on } [0,4]: \left(\frac{4}{128}\right)^3 \times \frac{127 \times 128 \times 255}{6} = \frac{127 \times 128 \times 255}{32{,}768 \times 6} = 21.0839843$$

$$\text{256 rectangles, on } [0,4]: \left(\frac{4}{256}\right)^3 \times \frac{255 \times 256 \times 511}{6} = \frac{255 \times 256 \times 511}{262{,}144 \times 6} = 21.2084961$$

Number of partitions	Rectangle sum
4	14
8	17.5
16	19.375
32	20.34375
64	20.8359375
128	21.0839843
256	21.2084961

The table and calculations above going up to 256 partitions suggest the rectangular sum will continue to get larger; as it does, more and more of the area under the curve will be included in the summation process. Does it seem as though the summation of the rectangles could ever be as much as 100? Could there ever be enough rectangles? The rectangles are getting more and more numerous, but each of their areas is getting smaller. That is what the process called "exhaustion" does. It "exhausts" more and more of the unused areas under the curve and the result is that the sum of rectangle areas is getting closer and closer to the area under the curve. See the figures on page 103.

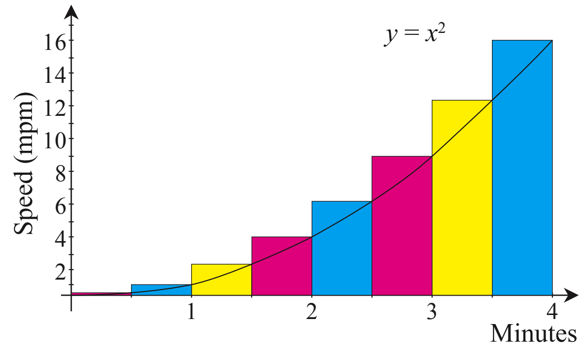
What would happen if we approached the area under the curve from the other direction, upper approximations getting closer and closer to the actual area? See the figure below. As before, the y-axis has been compressed for demonstration purposes.



x	Rectangle height x^2	Rectangle base $\frac{\text{interval}}{\text{number of rectangles}}$	Area of the rectangle $a = bh$
1	1	1	1
2	4	1	4
3	9	1	9
4	16	1	16

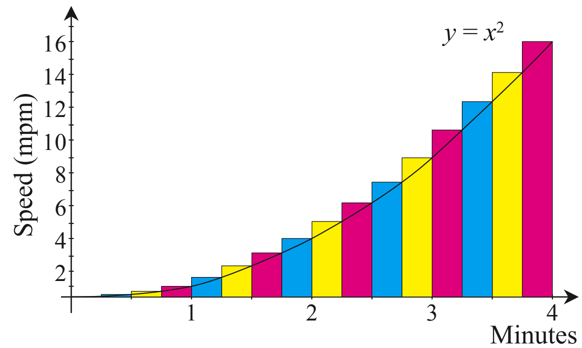
The sum of the four rectangles is $1 + 4 + 9 + 16 = 30$ units. The distance that the rocket traveled over four minutes is approximated to be 30 miles. Clearly this approximation is high as there are areas above the curve that were included in this approximation and should not have been.

Continuing this process for eight rectangles, we get the figure at the right.

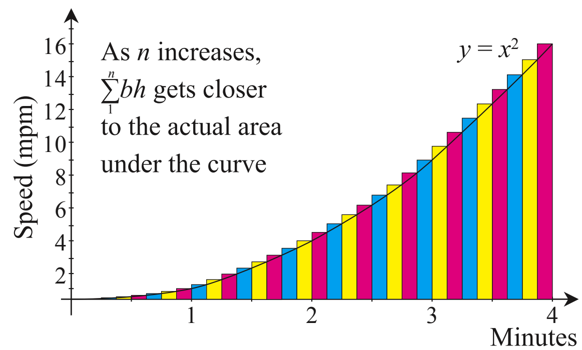


The sum of these eight rectangle areas is $\frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{16}{8} + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} + \frac{64}{8} = \frac{204}{8} = 25.5$ units. Therefore, the distance the rocket traveled over four minutes is approximately 25.5 miles.

x	Rectangle height x^2	Rectangle base $\frac{\text{interval}}{\text{number of rectangles}}$	Area of the rectangle $a = bh$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$
1	$1 = \frac{4}{4}$	$\frac{1}{2}$	$\frac{1}{2} = \frac{4}{8}$
$\frac{3}{2}$	$\frac{9}{4}$	$\frac{1}{2}$	$\frac{9}{8}$
2	$4 = \frac{16}{4}$	$\frac{1}{2}$	$2 = \frac{16}{8}$
$\frac{5}{2}$	$\frac{25}{4}$	$\frac{1}{2}$	$\frac{25}{8}$
3	$9 = \frac{36}{4}$	$\frac{1}{2}$	$\frac{9}{2} = \frac{36}{8}$
$\frac{7}{2}$	$\frac{49}{4}$	$\frac{1}{2}$	$\frac{49}{8}$
4	$16 = \frac{64}{4}$	$\frac{1}{2}$	$\frac{16}{2} = \frac{64}{8}$



Continuing this pattern for 16 and 32 rectangles, we get the figures at right.



Using the data from the table, let's sum up the areas of the graph with eight rectangles.

$$\text{Area} = \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{16}{8} + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} + \frac{64}{8} = \frac{204}{8} = 25.5$$

$$= \text{area}_1 + \text{area}_2 + \text{area}_3 + \text{area}_4 + \text{area}_5 + \text{area}_6 + \text{area}_7 + \text{area}_8 = \text{total area}$$

$$= b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4 + b_5h_5 + b_6h_6 + b_7h_7 + b_8h_8$$

Now, since the base of all these rectangles is the same—$\frac{\text{interval length}}{\text{number of rectangles}}$—substitute b for b_n:

$$\text{Area} = bh_1 + bh_2 + bh_3 + bh_4 + bh_5 + bh_6 + bh_7 + bh_0$$

$$= b \times (h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7 + h_0), \quad \text{factor out } b$$

$$= \frac{1}{2} \times \left[\left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{6}{2}\right)^2 + \left(\frac{7}{2}\right)^2 + \left(\frac{8}{2}\right)^2\right]$$

$$= \frac{1}{2} \times \left[\frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4} + \frac{25}{4} + \frac{36}{4} + \frac{49}{4} + \frac{64}{4}\right] = \frac{1}{2} \times \frac{1}{4} \times [1 + 4 + 9 + 16 + 25 + 36 + 49 + 64], \quad \text{factor out } \frac{1}{4}$$

$$= \frac{1}{8} \times \left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2\right] = \frac{1}{8} \times \text{the sum of the squares of the integers from 1 to 8}$$

$$= \frac{1}{8} \times \frac{8 \times 9 \times 17}{6} = 25.5, \quad \text{a formula from precal states } \sum_1^n i^2 = \frac{(n)(n+1)(2n+1)}{6}, \text{ see Appendix H}$$

In general, this work suggests that the area of any n rectangles could be found by the formula $b^3 \times \frac{(n)(n+1)(2n+1)}{6} = \left(\frac{\text{interval length}}{\text{number of rectangles}}\right)^3 \times \frac{(n)(n+1)(2n+1)}{6}$. Note that this only works to approximate the area under the curve $y = x^2$ on the interval from zero to four.

$$16 \text{ rectangles, on } [0,4]: \left(\frac{4}{16}\right)^3 \times \frac{16 \times 17 \times 33}{6} = \frac{16 \times 17 \times 33}{64 \times 6} = 23.375$$

$$32 \text{ rectangles, on } [0,4]: \left(\frac{4}{32}\right)^3 \times \frac{32 \times 33 \times 65}{6} = \frac{32 \times 33 \times 65}{512 \times 6} = 22.3437$$

$$64 \text{ rectangles, on } [0,4]: \left(\frac{4}{64}\right)^3 \times \frac{64 \times 65 \times 129}{6} = \frac{64 \times 65 \times 129}{4{,}096 \times 6} = 21.8359375$$

$$128 \text{ rectangles, on } [0,4]: \left(\frac{4}{128}\right)^3 \times \frac{128 \times 129 \times 257}{6} = \frac{128 \times 129 \times 257}{32{,}768 \times 6} = 21.5839844$$

$$256 \text{ rectangles, on } [0,4]: \left(\frac{4}{256}\right)^3 \times \frac{256 \times 257 \times 513}{6} = \frac{256 \times 257 \times 513}{262{,}144 \times 6} = 21.4584961$$

Number of partitions	Lower rectangle sum	Upper rectangle sum
4	14	30
8	17.5	25.5
16	19.375	23.375
32	20.34375	22.3437
64	20.8359375	21.8359375
128	21.0839843	21.5839844
256	21.2084961	21.4584961

Are you getting another one of those déjà vu feelings? Something seems familiar here.

Reorganizing the current area sum data from the table shown above as we did when approximating instantaneous speeds in the Chapter 1 review, we get the following table.

	Lower area rectangles							Upper area rectangles				
Number of rectangles	64	→	128	→	256	→	∞	←	256	←	128	←	64
Area sum of rectangles	20.83594	→	21.08398	→	21.208496	→	Actual area under curve	←	21.458496	←	21.58398	←	21.83594

Limit of an Infinite Series

Now it is more clear that the lower area rectangle sums are increasing toward the area under the curve $y = x^2$ from below while the upper area rectangle sums are decreasing toward the same value (area under the curve) from above. The upper and lower sums are approaching the same value—a *limit*. That limit is the area under the curve.

This is all very similar to the discussion and table in Chapter 1 when we showed secant-line slopes as they approached the slope of the tangent to the curve $y = x^2$ at $x = 8$ from both the right and the left. That tangent slope of 16 limited the progression of secant slopes from both the left and the right. Remember LimitMan in Chapter 1?

x	7.9	→	7.99	→	7.999	→	8.0	←	8.001	←	8.01	←	8.1
m	15.9	→	15.99	→	15.999	→	16.0???	←	16.001	←	16.01	←	16.1

Limit of an infinite Sequence of Secant Slopes

In Appendix B, we will see a new kind of limit, the limit of a function.

In order for the rectangles to perfectly fit under the curve they will all need to be *very skinny* and there will need to be a lot of them. Intuitively you can think of this situation as "infinite rectangles, each infinitesimally narrow."

	Lower area rectangles								Upper area rectangles				
Number of rectangles	64	→	128	→	256	→	∞	←	256	←	128	←	64
Area sum of rectangles	$\sum_{i=0}^{63} f(x_i)\Delta x$	→	$\sum_{i=0}^{127} f(x_i)\Delta x$	→	$\sum_{i=0}^{255} f(x_i)\Delta x$	→	Actual area under curve	←	$\sum_{i=1}^{256} f(x_i)\Delta x$	←	$\sum_{i=1}^{128} f(x_i)\Delta x$	←	$\sum_{i=1}^{64} f(x_i)\Delta x$

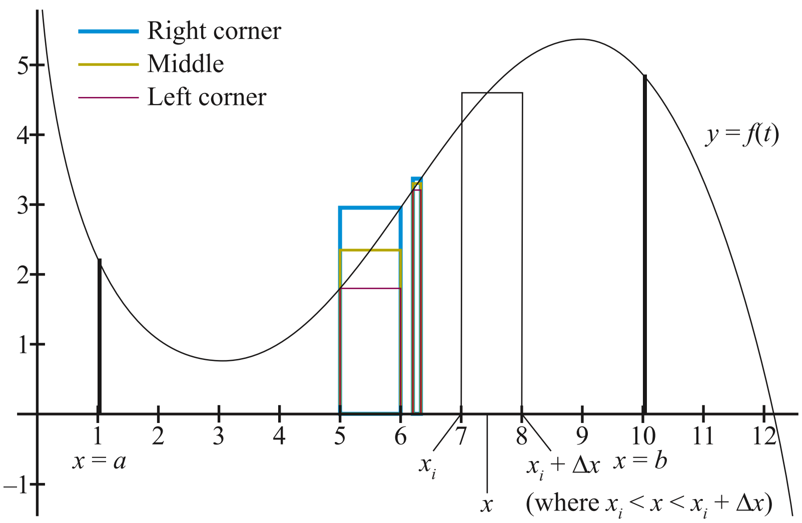
The lower area rectangle sums are increasing toward the area under the curve $y = f(x)$ from below while the upper area rectangle sums are decreasing toward the same value (area under the curve) from above. The math symbolism used when that happens is

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} l_i\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} u_i\Delta x,$$

where l_i represents the lengths of all rectangles "lower than" the curve and u_i represents the lengths of all the "upper" rectangles, Δx represents the width of each rectangle in each set and is determined by the expression $\frac{\text{upper domain}-\text{lower domain}}{\text{number of partitions}}$, n is the number of rectangles being summed in this set, and i is a counter that keeps track of which rectangle you are summing.

There is something to keep in mind. It can be demonstrated that when $n \to \infty$, it really does not matter too much whether the area is approached using lower sums or using upper sums or rectangles that are partly above and partly below. To see this, note that the height of the rectangle *is* significantly affected by this choice when the width of the rectangle is large, but it *is not* significantly affected by this choice when the width of the rectangle is small. The figure at right can help clarify this point.

All this implies that you can choose any x value in the ith subinterval and that choice will not affect the limit (sum of areas) when $n \to \infty$.



The lower and upper sum limits,

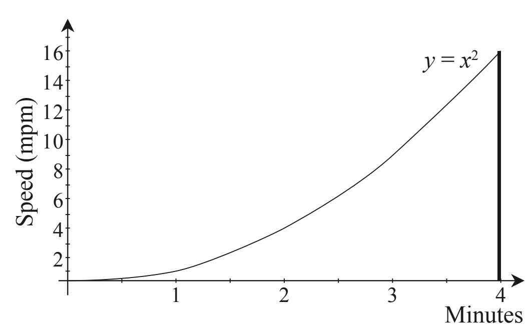
$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(l_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(u_i)\Delta x,$$

can be seen in the definition at right.

> Definition of Area Under a Curve
>
> Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by f, the x-axis, and the vertical lines $x = a$ and $x = b$, is
>
> $$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x,$$
>
> where $a < x_i < b$, with i designating the subinterval of x and $\Delta x = \frac{b-a}{n}$.
> *Calculus with Analytic Geometry* (5th ed.) Larson, Hostetler, and Edwards, D.C. Heath, 1994, pg. 265, modified by this author.

Let's try this new definition out on the problem in Chapter 11!

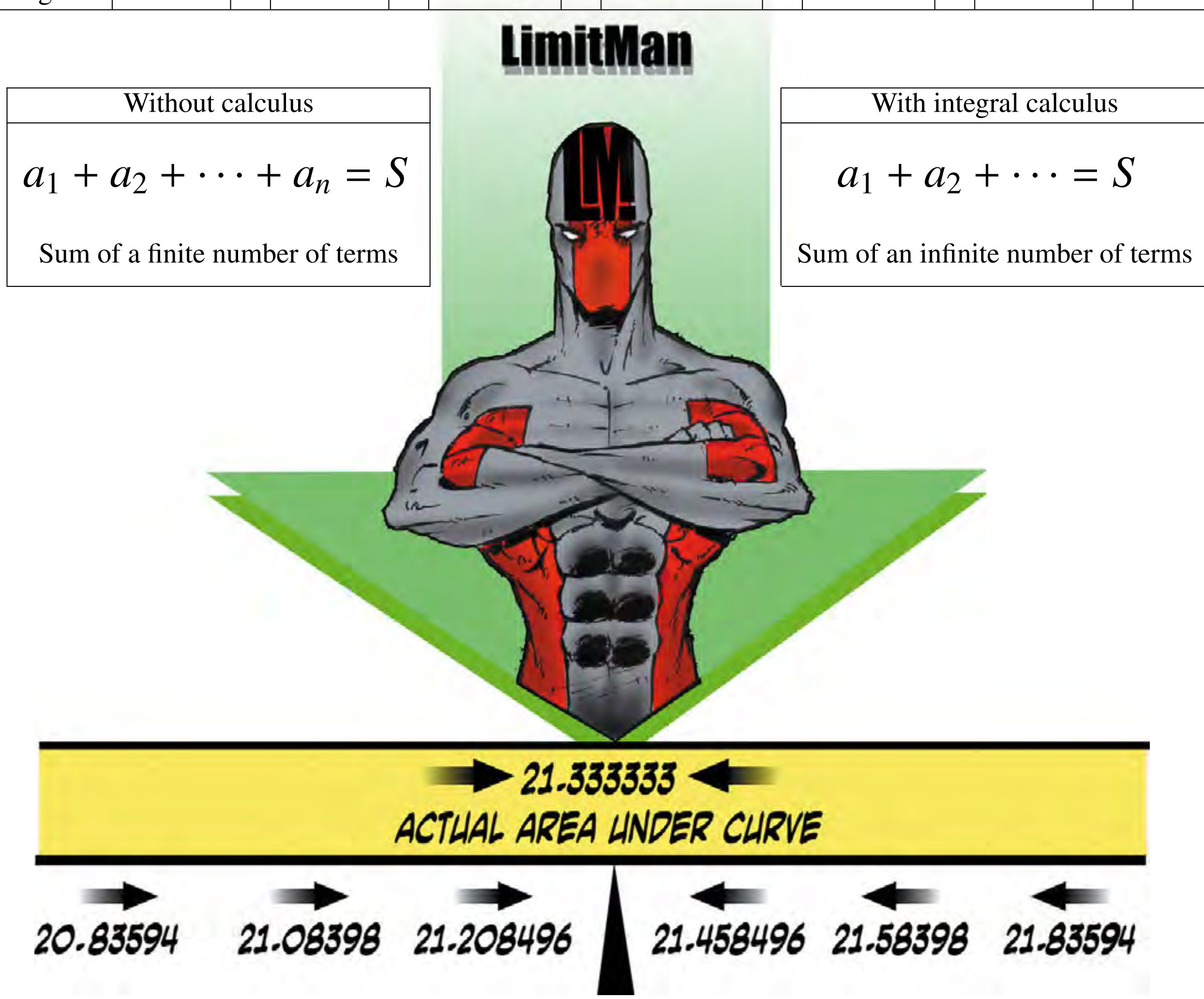
> A rocket is accelerating (speeding up). Its speed (in mpm) is given by the function $r = t^2$. *How far did the rocket travel in the first 4 minutes?*



$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\left[\left(\frac{\text{interval length}}{\text{num rectangles}}\right)^3 \frac{n(n+1)(2n+1)}{6}\right], \quad \text{from work done in Chapter 11}\\
&= \lim_{n\to\infty}\left[\left(\frac{4}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty}\left[\frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}\right], \quad \text{because } \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\\
&= \lim_{n\to\infty}\left[\frac{64}{n^2}\frac{(n+1)(2n+1)}{6}\right], \quad \text{canceling out the } n \text{ terms}\\
&= \lim_{n\to\infty}\left[\frac{32}{n^2}\frac{2n^2+n+2n+1}{3}\right], \quad \text{reducing } \frac{64}{6} \text{ to } \frac{32}{3} \text{ and FOIL}\\
&= \lim_{n\to\infty}\left[\frac{32}{n^2}\frac{2n^2+3n+1}{3}\right] = \lim_{n\to\infty}\left[\frac{64n^2+96n+32}{3n^2}\right]\\
&= \lim_{n\to\infty}\frac{64n^2}{3n^2} + \lim_{n\to\infty}\frac{96n}{3n^2} + \lim_{n\to\infty}\frac{32}{3n^2}, \quad \text{limit of a sum is the sum of the its limits}\\
&= \lim_{n\to\infty}\frac{64}{3} + 0 + 0, \quad \text{canceling } \frac{n^2}{n^2} \text{ and reducing } \frac{96n}{3n^2}\text{, then the denominators go to } \infty\\
&= 21.333333, \quad \text{actual area under the curve!}
\end{aligned}
$$

Let’s compare this answer with the lower and upper sum approximations we obtained back in Chapter 11 when we were doing this same problem. As the number of rectangles goes to infinity, the sum of areas goes to the area under the curve.

	Lower area rectangles								Upper area rectangles			
Number of rectangles	64	→	128	→	256	→	∞	←	256	←	128	←	64
Area sum of rectangles	20.83594	→	21.08398	→	21.208496	→	21.333333	←	21.458496		21.58398	←	21.83594



Chapter 12 Review

Definition of $f'(x)$

$$f'(x) = \lim_{a \to x} \frac{f(x) - f(a)}{x - a}$$

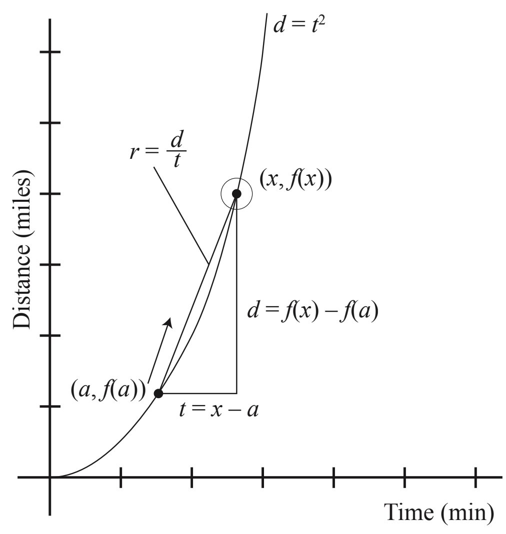
The domain of $f'(x)$ is the set consisting of every number, x, at which the above limit exists.

Johnson and Kiokemeister's Calculus with Analytic Geometry (5th ed.), Johnson, Kiokemeister, and Wolk, Allyn and Bacon, 1974, pg. 91, modified by this author.

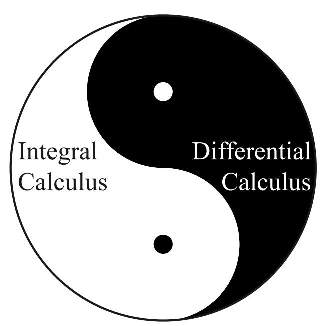
In Chapter 12, attempts to determine the area under a curve using successive approximation of finite rectangle areas are replaced with the summation of infinite rectangle areas allowing us to find the area under a curve exactly rather than approximately. Just as in Chapter 4, where the Definition of $f'(x)$ allowed us to find an exact value for the slope of a tangent to a curve, the new definition of the area under a curve,

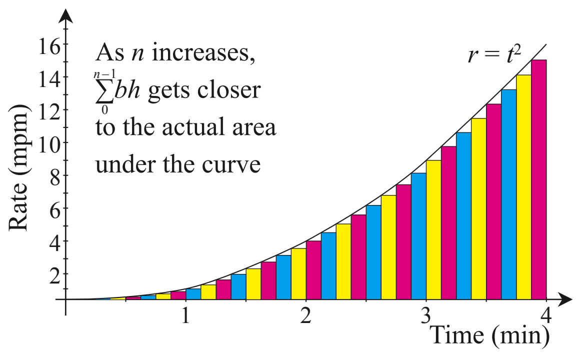
> Definition of Area Under a Curve
>
> Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by f, the x-axis, and the vertical lines $x = a$ and $x = b$, is
>
> $$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$
>
> where $a < x_i < b$, with i designating the subinterval of x and $\Delta x = \frac{b-a}{n}$.
> *Calculus with Analytic Geometry* (5th ed.) Larson, Hostetler, and Edwards, D.C. Heath, 1994, pg. 265, modified by this author.

allows us to find an exact value for the area under a curve. This new definition was demonstrated by applying it to the problem that was worked back in Chapter 11. Intuitively, the "Definition of Area Under a Curve" can be thought of as summing up the areas of infinitesimally thin rectangles. Both of these definitions involve the idea of "limit." In this book, we see three kinds of limits: limit of a sequence of secant slopes, limit of a series, and limit of a function.



"A missile is accelerating. Its distance is given by the function $d = t^2$. *Find the instantaneous speed of the missile* when it strikes its target at $t = 4$." **The definition of $f'(x)$ is applied here to find the limit of an infinite sequence of ratios.** (Recall that $d = rt$ so $r = d/t$.) As $n \to \infty$, $\frac{d_1}{t_1}, \frac{d_2}{t_2}, \frac{d_3}{t_3}, \frac{d_4}{t_4} \ldots \to r$, the instantaneous speed at t.

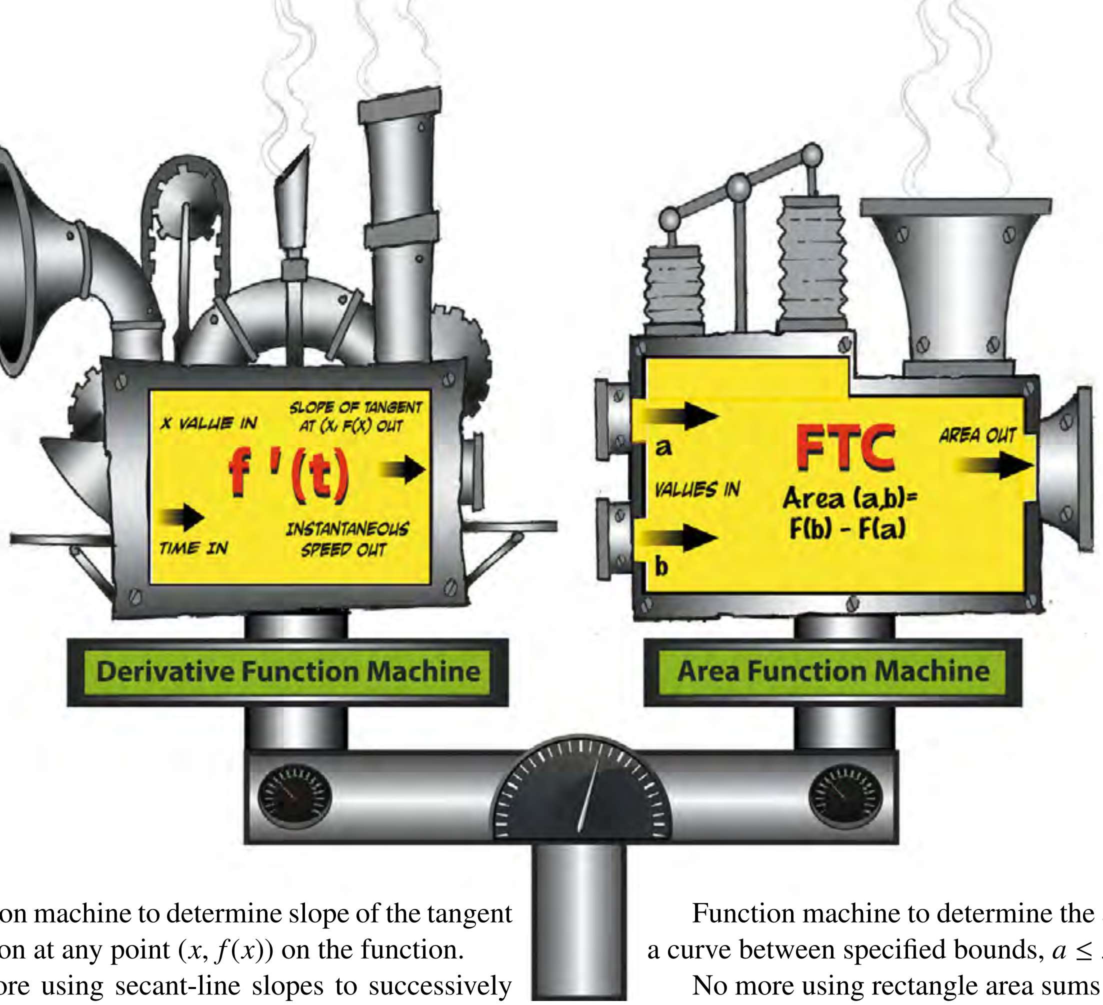




"A missile is accelerating. Its speed is given by the function $r = t^2$. *Find the distance the missile traveled at $t = 4$.*" **The definition of "area under a curve" is applied here on the same three variables as were used in the problem above (d, r, and t) to find the limit of an infinite series of products.** As $n \to \infty$, $r_1t_1 + r_2t_2 + r_3t_3 + r_4t_4 + \cdots \to d$. (Recall from pg 103 that $r \times t = \frac{\text{miles}}{\cancel{\text{min}}} \times \cancel{\text{min}} = \text{miles}$.) The definition of f' generates an infinite "sequence," while the definition of area under a curve generates an infinite "series."

Since calculating the area under a curve is mathematically equivalent to solving many applied problems, this skill takes on new importance and significance. Finding the area under the curve $y = x^2$ was easy because that particular problem involved summing up perfect squares, $\sum_{i=1}^{n} i^2$. A clever formula (Appendix H) was used to develop another formula, $\frac{\text{interval length}}{\text{number of rectangles}} \frac{n(n+1)(2n+1)}{6}$. In general, the functions we will be using will not lend themselves to the development of such clever formulas. Even if they did, there is an infinity of function possibilities, and it would not be desirable to have to develop a special formula for each of them. What we need is a generic approach that will work for all functions. That is what Chapter 13, and the Fundamental Theorem of Calculus, is about.

The goal of the Fundamental Theorem of Calculus is to develop a function that will allow a person to obtain the area under a curve simply by substituting values a and b into that function. This function, Area(a, b), will be analogous to the previously discussed $f'(x)$ that allows us to quickly and simply evaluate the instantaneous speed or slope of a tangent to a polynomial curve at a specified point $(x, f(x))$.



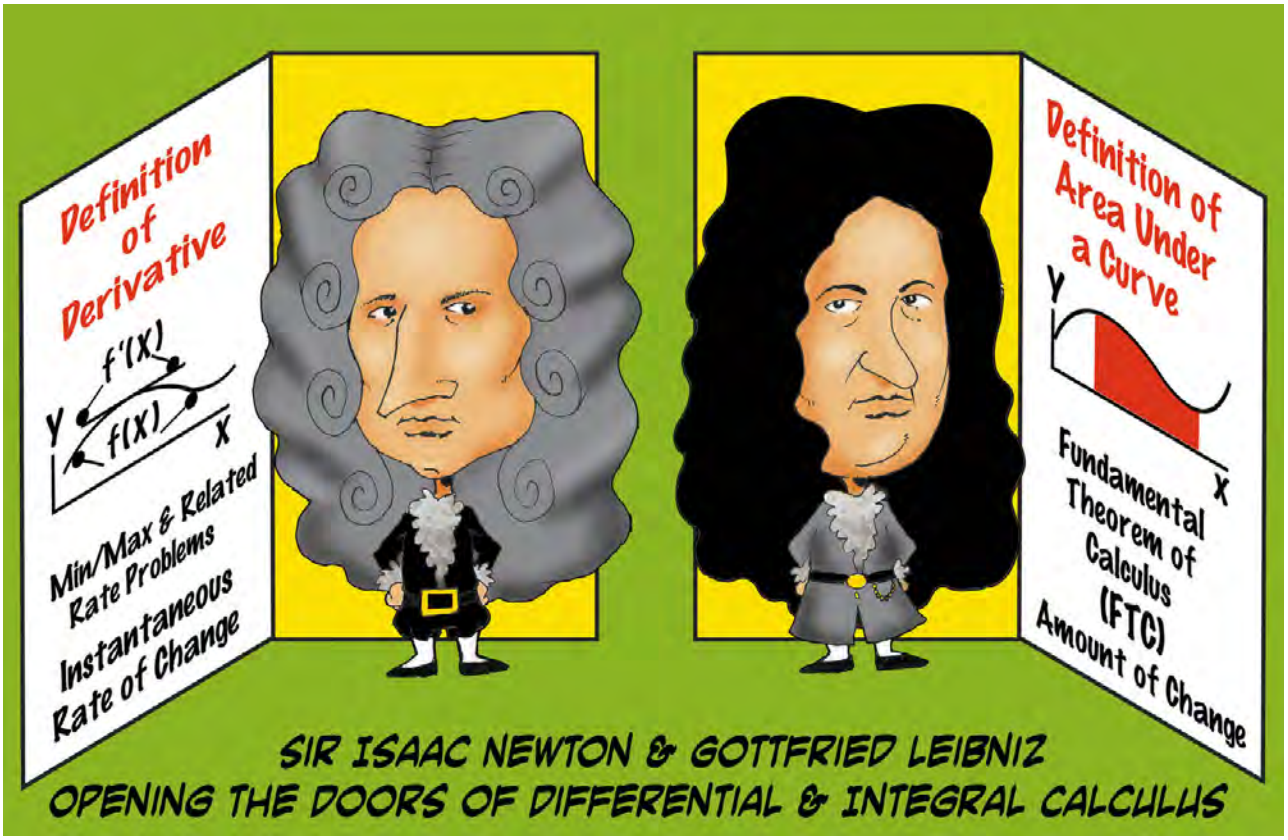
Function machine to determine slope of the tangent to a function at any point $(x, f(x))$ on the function.

No more using secant-line slopes to successively approximate the slope of a tangent.

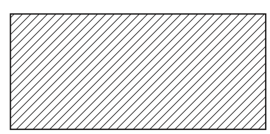
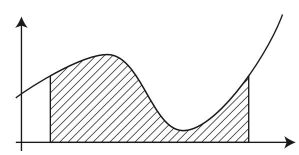
Function machine to determine the area under a curve between specified bounds, $a \le x \le b$.

No more using rectangle area sums to successively approximate the area under a curve.

Before learning more about the function Area(a, b) for $f(x)$, it will be instructive to talk about a new kind of function called an "antiderivative function." If $f(x)$ is some function, we know that the derivative can be written as $f'(x) = \frac{d}{dx}[f(x)]$. Similarly, we write the antiderivative of $f(x)$ as $F(x) = \int f(x)dx$. You have seen, in both arithmetic and algebra, ideas that were somewhat similar. When you were taught $3+1 = 4$, therefore $3 = 4-1$, the term "inverse operation" was used. Similarly $5 \times 2 = 10$ can be written as $5 = \frac{10}{2}$ because multiplication and division are inverse operations. In algebra, you were taught that if $f(x) = 2x + 1$ and $g(x) = \frac{x-1}{2}$ then $f[g(x)] = x = g[f(x)]$ because $f(x)$ and $g(x)$ are inverse functions. The derivative and antiderivative functions are inverse functions: $\int f'(u)du = f(u) = \frac{d}{du}[F(u)]$.



Before Newton and Leibniz, a branch of mathematics known as Euclidean geometry was the consistent synergism of undefined terms (points, lines, space, etc.), common notions (postulates), and theorems. After Newton and Leibniz, calculus was to be the synergism of basic algebraic properties (commutative, associative, distributive, etc.), definitions (definition of derivative, definition of area under a curve), and easy-to-apply theorems (derivative of x^n; derivative of a polynomial sum, difference, product, and quotient; derivative of trigonometric functions; FTC) which derived from the properties and definitions. As with geometry, all the properties, definitions, and theorems in calculus would be consistent (not contradict each other).

Without calculus	With integral calculus
Area of a rectangle	Area under a curve
	

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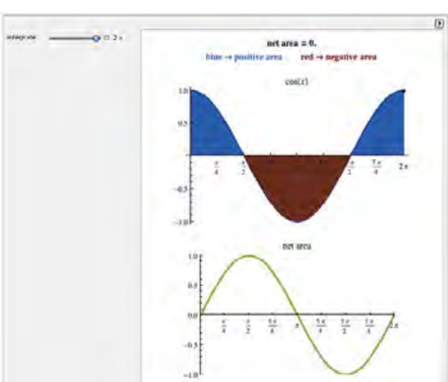
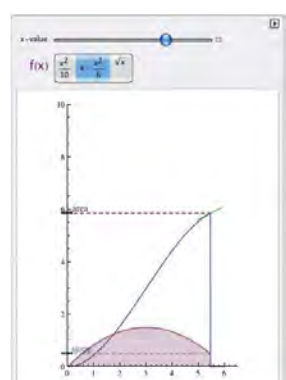


The Fundamental Theorem of Calculus
http://demonstrations.wolfram.com/FundamentalTheoremOfCalculus



The Fundamental Theorem of Calculus
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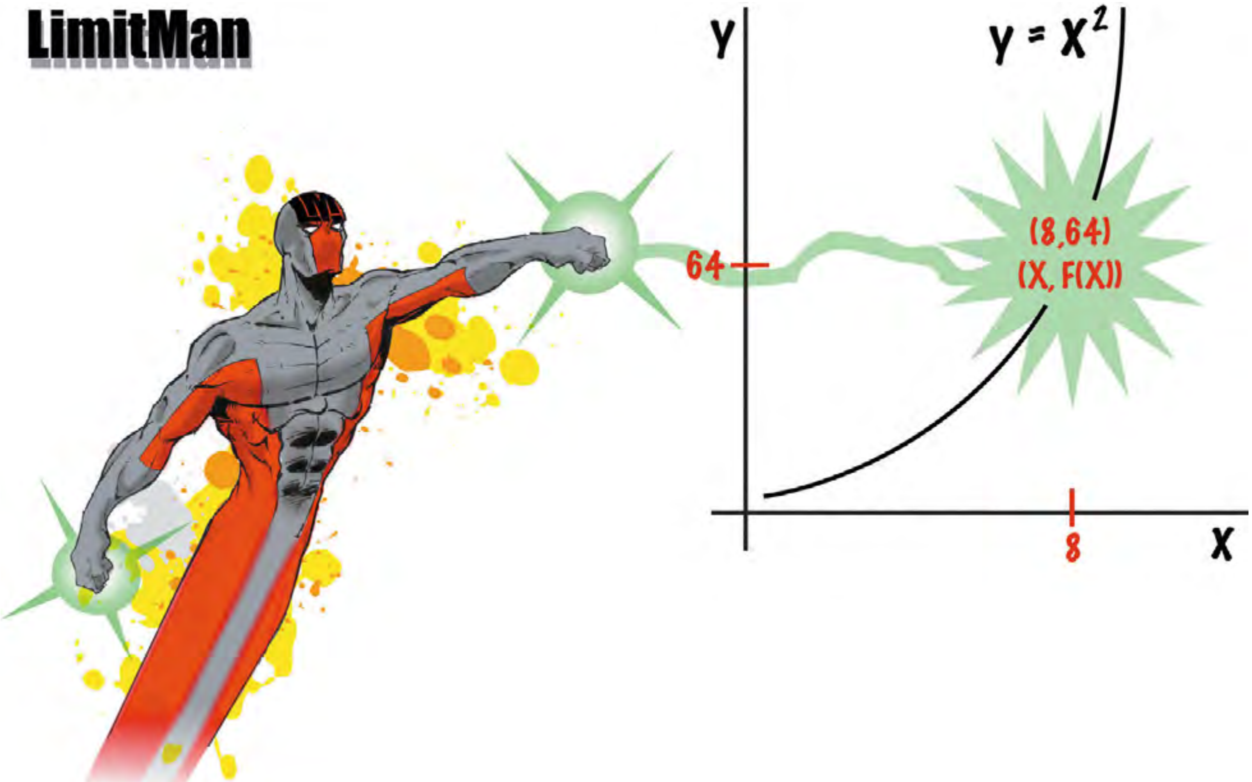
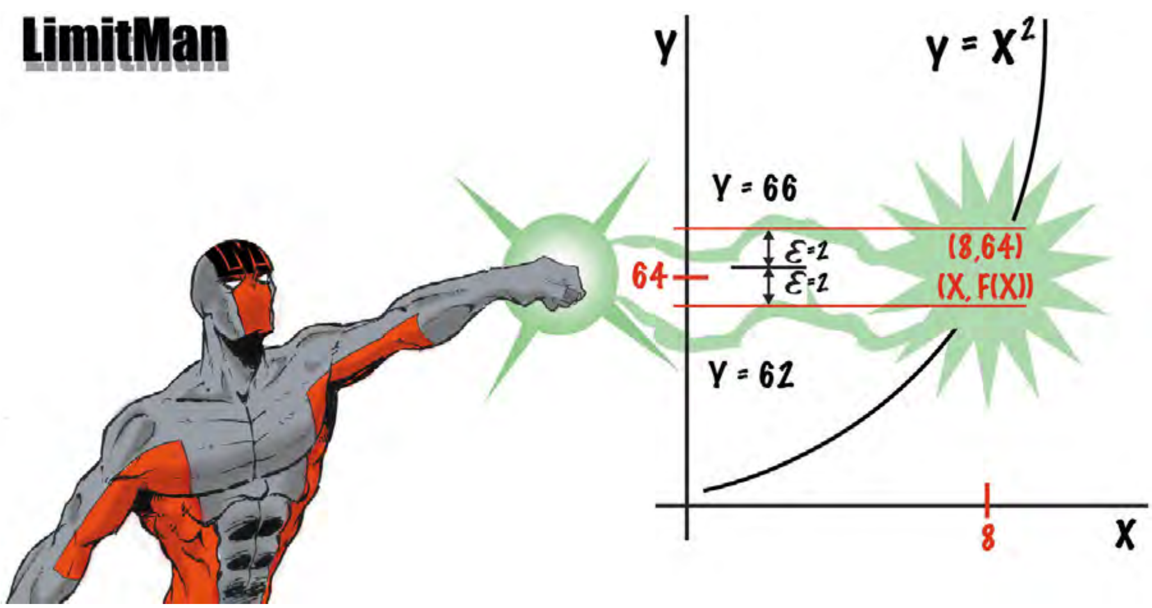
It's time for the LimitMan — Piggy Challenge Game!!!

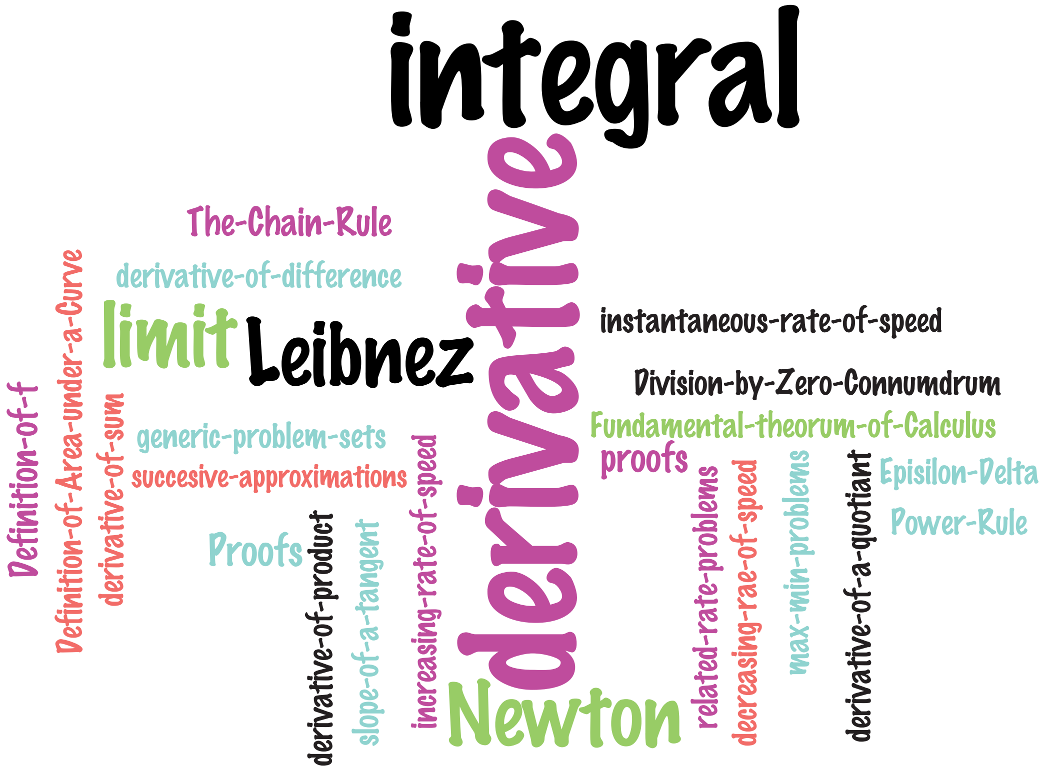
LimitMan is looking at the point (8, 64) on the function $f(x) = x^2$. He wants to know if there is a "limit" or barrier there when $x = 8$. That is, "Do the y values approaching from both the left and the right approach the same value (64) … a.k.a. the limit of $f(x) = x^2$ at 8?" We have seen tables in Chapter 2 that there is a limit to secant line-slopes when $x = 8$. What would be considered proof that a limit exists at $x = 8$ on the function $f(x) = x^2$?



LimitMan chooses the value $\epsilon = 2$ and uses it to create an interval 62 $(64 - \epsilon)$ to 66 $(64 + \epsilon)$ for the range values of the function.



Twenty Key Ideas in Beginning Calculus was conceived when the author noticed that many high school AP programs, especially English, often required summer reading for their students. Some math programs have been known to experiment with requiring the reading of historical books about mathematics or famous mathematicians, but they do not get much curricular payback for their students' time. Meaningful, accessible, materials that could be assigned to students for summer reading and that would support the calculus curriculum simply do not exist. Such materials sound like an oxymoron. Common wisdom has it that calculus materials are inherently too difficult for students to read and study on their own. Any attempt to create such materials would fail.



Twenty Key Ideas in Beginning Calculus does not claim or intend to be a calculus text. It is a creative sequencing and presentation of a subset of topics in the standard calculus curriculum. The author makes heavy use of "anticipatory sets," "schedules of reinforcement," "connection of mathematical ideas," "connection to real-world applications," pacing, patterns, visuals, examples and counterexamples, evolution and organization of ideas, repeated threading and spiraling of concepts, and especially repetition, repetition, and repetition. Four of the fourteen chapters (1, 2, 3, & 11) are written using only introductory algebra skills to introduce both calculus vocabulary and concepts. Limits and other major calculus concepts are taught intuitively using tables and visuals. All major proofs are relegated to the appendices, allowing students to customize their learning experience according to their ability and interest for rigor.

The author, **Dan Umbarger**, has taught various levels of mathematics from grade 5 to grade 12 for over 30 years. He currently teaches AP Computer Science at a Dallas area "majority minority" school. He is married and the proud father of three children: Jimmy, Terri, and Keelan. He is also the author of *Explaining Logarithms* and "Explaining Bayes Theorem" at www.mathlogarithms.com.

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